Chapter 4

Continuity of Functions

Open sets, closed sets and related notions of sets in $\mathbb{R}^n$ are introduced in Section 1. Limits and continuity of functions of several variables are discussed in Sections 2 and 3 respectively. The study is parallel to that of functions of a single variable.

Caution: Beginning from this chapter, a generic point in $\mathbb{R}^n$ will be simply written as x, y, u, v, etc rather than in bold letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$, etc.

4.1 Point Sets in $\mathbb{R}^n$

In this section we define some notions concerning sets in $\mathbb{R}^n$. Although they will become fundamental in more advanced courses, these notions are introduced here mainly for the ease of formulation of many subsequent results. We will study them thoroughly in MATH3060 Mathematical Analysis III and MATH3070 Introductory Topology.

To kick off, we denote the n-dimensional open ball by

$$B_r(x) = \{y \in \mathbb{R}^n : \ |y - x| < r\} ,$$

and the closed ball by

$$\overline{B_r(x)} = \{y \in \mathbb{R}^n : \ |y - x| \le r\} .$$

Whenever convenient, $B_r(x)$ will be written as B_r, $B(x)$, or even B. Let E be a non-empty set in $\mathbb{R}^n$. A point $x \in E$ is called an **interior point** of E if there exists some $r > 0$ such that $B_r(x) \subset E$. Roughly speaking, an interior point of E is a point in E that is surrounded by points in E. A set is called an **open set** if it consists of interior points. We define the empty set ϕ to be an open set. The complement of an open set is called a **closed set**. Be cautious that there are sets which are open and closed at the same time. Indeed, the entire space $\mathbb{R}^n$ is open as it contains all open balls and on the other hand closed as it is the complement of the empty set. It follows that the empty set is also closed as it is the complement of $\mathbb{R}^n$. One can show that these two sets are the only open and closed sets in $\mathbb{R}^n$, but we do not need this fact.

A point $x \in \mathbb{R}^n$ is called a **boundary point** of the set E if for each $r > 0$, both intersections $B_r(x) \cap E$ and $B_r(x) \cap \mathcal{C}E$ where $\mathcal{C}E$ is the complement of E are nonempty. As $B_r(x) \subset B_s(x), r < s$, it suffices to have a sequence $\{r_n\} \to 0$ to fulfil this requirement. A boundary point of E may or may not belong to E, but is always surrounded by points inside and outside E. The collection of all boundary points of the set E forms the **boundary** of E and is denoted by ∂E.

Example 4.1. The ball $B_r(x)$ is an open set. For, let $y \in B_r(x)$. When $y \neq x, 0 < |y-x| < r$. Setting $\rho = r - |y-x| > 0$, $B_\rho(y) \subset B_r(x)$. When $y = x$, simply observe that $B_r(x) \subset B_r(x)$. By a similar argument, the exterior of the ball $\mathcal{C}\overline{B_r(x)} = \{y : \ |y-x| > r\}$ is also open. It follows from definition that the closed ball $\overline{B_r(x)}$ is closed. Observing that the union of open sets are still open, the sphere $S = \{y : |y-x| = r\}$ which is the complement of $B_r(x) \cup \mathcal{C}\overline{B_r(x)}$ is closed too.

Example 4.2. The intervals

$$(a, b), \quad (-\infty, b), \quad (a, \infty), \quad a, b \in \mathbb{R},$$

are open sets in $\mathbb{R}$. So are their unions. The intervals

$$\{a\} = [a, a], \quad [a, b], \quad (-\infty, b], \quad [a, \infty) \ ,$$

are closed sets in $\mathbb{R}$. So are their intersections. The boundaries of $(a, b), [a, b], [a, b)$ and $(a, b]$ are $\{a, b\}$. In particular, the the boundary of the singleton $\{a\}$ is $\{a\}$ itself. It has no interior points.

Example 4.3. Let D be the unit disk in $\mathbb{R}^2$, that is, $D = \{(x, y) : \ x^2 + y^2 < 1\}$ and S the unit circle $S = \{(x, y) : \ x^2 + y^2 = 1\}$. Consider

(a) $E_1 = D$,

(b) $E_2 = D \cup S$,

(c) $E_3 = D \cup (S \cap \{(x, y) : x > 0\})$,

(d) $E_4 = D \cap \{(x, y) : \ x, y \in \mathbb{Q}\}$.

Then E_1 is open, E_2 is closed, E_3 and E_4 are neither open nor closed. The boundary of E_1, E_2, E_3 is S and the boundary of E_4 is $D \cup S$.

The following result explains why a closed set is named closed.

Theorem 4.1. *A set in $\mathbb{R}^n$ is closed if and only if it contains all its boundary points.*

Proof. * $\Rightarrow$) Let E be a closed set and $x \in \partial E$. If $x \in \mathcal{C}E$, as $\mathcal{C}E$ is open, $B_r(x) \subset \mathcal{C}E$ for some $r > 0$ which shows that x cannot be a boundary point of E, contradiction holds. Hence x must belong to E.

$\Leftarrow$) If E is not closed, then $\mathcal{C}E$ is not open. We can find some $z \in \mathcal{C}E$ such that $B_r(z) \cap E \neq \phi$ for all $r > 0$. As $B_r(z) \cap \mathcal{C}E = \{z\}$ is never empty, z is a boundary point of E. We have found a boundary point of E lying outside E. $\square$

We may reformulate this proposition in terms of sequences. That is, a nonempty set E is closed if and only if for every $\{x_n\}, x_n \in E$, converging to some $x \in \mathbb{R}^n$, $x \in E$. It is a good exercise to provide a proof of it.

Theorem 4.2. *For every set E in $\mathbb{E}^n$, $E \cup \partial E$ is a closed set.*

Proof. * First we claim that $\partial(\partial E) \subset \partial E$, that is, any boundary point of ∂E is again a boundary point of E. Let z be a boundary point of ∂E. Then $B_r(z) \cap \partial E \neq \phi$ for all $r > 0$. Pick a point $w \in B_r(z) \cap \partial E$. Since w is a boundary point of E and $B_r(z)$ is open, there is some $B_\rho(w) \subset B_r(z)$ such that $B_\rho(w) \cap E \neq \phi$ and $B_\rho(w) \cap \mathcal{C}E \neq \phi$. It follows that $B_r(z) \cap E \neq \phi$ and $B_r(z) \cap \mathcal{C}E \neq \phi$, that is, $z \in \partial E$.

Now we prove the proposition by showing that the complement of $E \cup \partial E$ is open. Let $z \in \mathcal{C}(E \cup \partial E)$. We need to find a ball $B_\rho(z) \subset \mathcal{C}(E \cup \partial E)$ for some $\rho > 0$. Suppose on the contrary this is not true, there is $\rho_n \to 0$ such that $B_{\rho_n}(z) \cap (E \cup \partial E) \neq \phi$. If $B_{\rho_n}(z) \cap E \neq \phi$ for infinitely many n, z is a boundary point of E, contradiction holds. Therefore, $B_{\rho_n}(z) \cap \partial E \neq \phi$ for all large n, which implies $z \in \partial(\partial E)$, but $\partial(\partial E) \subset \partial E$. We have arrived at the same contradiction again. $\square$

The **closure** of a set E is defined to be $E \cup \partial E$ and denoted by $\overline{E}$. It is the smallest closed set containing E. Again it is a good exercise to prove it.

4.2 Limits of Functions

In advanced calculus we are mainly concerned with functions defined on subsets of the Euclidean space. Let E be a non-empty set in $\mathbb{R}^n$. Recall that a real-valued function in E is a rule to assign every point in E a unique real number. Whenever E and the function are specified, E is called the domain of definition of the function.

Consider a function f whose domain of definition is $E \subset \mathbb{R}^n$. Let x be an interior point or a boundary point of E so that $B_r(x) \setminus \{x\} \cap E \neq \phi$ for all $r > 0$. The function f is said to have a **limit** at x if there exists a real number a such that, for each $\varepsilon > 0$,

$$|f(y) - a| < \varepsilon, \quad \forall y \in B_\delta(x),\ y \neq x\ ,$$

for some $\delta > 0$. Alternatively, you may write

$$|f(y) - a| < \varepsilon, \quad \forall y,\ 0 < |y - x| < \delta\ .$$

When this happens, we write

$$\lim_{y \to x} f(y) = a\ , \qquad \text{or} \qquad f(x) \to a \text{ as } y \to x\ .$$

We note that in the definition x may or may not belong to E, it is good as long as $B_r(x) \setminus \{x\}$ intersects E for every $r > 0$. Furthermore, whether or not f is defined at x does not matter. We are solely concerned with the limiting behavior of the function when x is approached.

The limit of a function can also be described in terms of sequences. A sequence $\{x_k\}$ in $\mathbb{R}^n$ is said to **converge** to x if for each $\varepsilon > 0$, there is some k_0 such that $|x_k - x| < \varepsilon$ for all $k \geq k_0$. When this happens, we use the notation $x_k \to x,\ k \to \infty$, or simply $x_k \to x$. The definition is the same as the one dimensional case, the only difference being the absolute value is now replaced by the Euclidean distance.

Proposition 4.3 (**Sequential Criterion for Limit**). *Let f be defined in E and $x \in \overline{E}$. The followings are equivalent:*

(a)

$$\lim_{y \to x} f(x) = a\ ,$$

(b)

$$f(x_k) \to a \textit{ whenever } x_k \to x, x_k \in E, x_k \neq x.$$

Proof. * $\Rightarrow$) Let $\{x_k\}$ be a sequence in E converging to x and $x_k \neq x$ for all n. For $\varepsilon > 0$, there is some δ such that $|f(y) - a| < \varepsilon$ for all $y \in E, y \neq x$. For this δ, there is some k_0 such that $|x_k - x| < \delta$ for all $k \geq k_0$. It follows that $|f(x_k) - a| < \varepsilon$ for all $k \geq k_0$, that is, $f(x_k) \to a$ as $k \to \infty$.

$\Leftarrow$) Assume on the contrary that $f(y)$ does not converge to a. There exists some $\varepsilon_0 > 0$ such that for each δ there corresponds some y_δ satisfying $|y_\delta - x| < \delta$ but $|f(y_\delta) - a| \geq \varepsilon_0$. Taking $\delta = 1/k$ and $x_k = y_{1/k}$ we see that the sequence $\{x_k\}$ converges to x but $|f(x_k) - a| \geq \varepsilon_0$. $\square$

The following basic result, usually called the limit theorem, has been discussed in the single variable case. It extends readily to the present situation.

Theorem 4.4 (**Limit Theorem**). *Let f and g be defined in $E \subset \mathbb{R}^n$ and $x \in \overline{E}$. Suppose that*

$$\lim_{y\to x} f(y) = a\ , \quad \lim_{y\to x} g(x) = b\ .$$

Then

(a)

$$\lim_{y\to x} \big(\alpha f(y) + \beta g(y)\big) = \alpha a + \beta b\ , \quad \alpha, \beta \in \mathbb{R},$$

(b)

$$\lim_{y\to x} f(y)g(y) = ab\ .$$

(c)

$$\lim_{y\to x} \frac{f(y)}{g(y)} = \frac{a}{b}\ ,$$

provided $b \neq 0$.

The following Sandwich Rule is a common tool in proving the existence of limits. It can be proved as in the single variable case.

Theorem 4.5 (**Sandwich Rule**). *Let f, g and h be defined in $E \subset \mathbb{R}^n$ and $x \in \overline{E}$. Suppose that*

(a) $h \le f \le g$ *in* E, *and*

(b) $\lim_{y\to x} h(y) = \lim_{y\to x} g(y) = a$.

Then $\lim_{y\to x} f(y)$ *exists and equals to* a.

It is extremely useful to know that the existence of limit is preserved under compositions with continuous maps.

Theorem 4.6. *Let f be defined in $E \subset \mathbb{R}^n$ and $x \in \overline{E}$. Suppose that*

$$\lim_{y\to x} f(y) = a\ .$$

Let Φ be a real-valued function defined on some open interval containing a and $\lim_{u\to a} \Phi(u) = c$. *Then*

$$\lim_{y\to x} \Phi(f(x)) = c\ .$$

Proof. Let $\{x_k\}$ be a sequence in $E \setminus \{x\}$ converging to x. For all large k, $f(x_k)$ is well-defined. Then $f(x_k)$ is a sequence converging to a. By the sequential criterion $\Phi(f(x_k)) \to c$, the theorem follows.

□

Example 4.4. Consider the function

$$f(x,y) = \frac{xy}{x^2+y^2},$$

which is defined in $\mathbb{R}^2/\{(0,0)\}$. We claim that it does not have a limit at the origin. For, taking $\{(x_k,0)\} \to (0,0)$, we see that $f(x_k,0)$ is equal to 0 constantly. Therefore, were the limit exist, it must be equal to 0. However, if we only $(x_k,y_k), x_k = y_k$, then $f(x_k,x_k) = 1/2$ which is never equal to 0. We conclude that this function does not have a limit at the origin. Indeed, in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, f becomes

$$f(x,y) = \cos\theta\sin\theta = \frac{1}{2}\sin 2\theta\ .$$

When the angle θ runs from 0 to 2π, the values of f cover the interval $[-1,1]$. How can it approach to a definite value?

Example 4.5. Consider the function

$$g(x,y) = \frac{xy^2}{x^2+y^2}\ , \quad (x,y) \neq (0,0)\ .$$

We claim that its limit at the origin exists and is equal to 0. For, we have

$$\left|\frac{xy}{x^2+y^2}\right| \leq \frac{1}{2}\ ,$$

it follows that

$$0 \leq |g(x,y)| \leq \frac{|y|}{2}\ .$$

By the Sandwich Rule

$$\lim_{(x,y)\to(0,0)} g(x,y) = 0\ .$$

For a function f defined near a point $x \in \mathbb{R}^n$ we can talk about iterated limit. Let us restrict to $n = 2$ for simplicity. Let (x_0,y_0) be an interior point of E over which a function f is defined. It makes sense to talk about the limit

$$\lim_{y\to y_0}\lim_{x\to x_0} f(x,y) \text{ and } \lim_{x\to x_0}\lim_{y\to y_0} f(x,y)\ .$$

However, even if these limits exist and are equal, it does not mean

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

exists. To illustrate this fact, look at the function in Example 4.2. It is easy to see that

$$\lim_{y\to 0}\lim_{x\to 0}\frac{xy}{x^2+y^2} \quad \text{and} \quad \lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2}\ ,$$

both exist and equal to 0, but

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$$

does not exist. One must stay sober when working on analysis!

4.3 Continuity of Functions

Continuity of functions is closely related to the existence of limit. Let f be a real-valued function defined on E and $x \in E$. It is continuous at x if (a) $\lim_{y\to x} f(y)$ exists and (b) the limit is equal to $f(x)$. In other words, f is **continuous** at x if and only if for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$|f(y)-f(x)| < \varepsilon, \quad \forall y \in E, \quad |y-x| < \delta\ .$$

We translate Proposition 4.3, Theorems 4.4 and 4.6 to three results concerning continuity.

Theorem 4.7 (**Sequential Criterion for Continuity**). *Let f be defined in E and $x \in E$. Then f is continuous at x if and only if for every $\{x_k\} \subset E$, $x_k \to x$ as $k \to \infty$,*

$$\lim_{k\to\infty} f(x_k) = f(x)\ .$$

Theorem 4.8. *Let f and g be defined in E and continuous at $x \in E$. Then for $\alpha, \beta \in \mathbb{R}$,*

(a) $\alpha f + \beta g$ is continuous at x,

(b) fg is continuous at x,

(c) f/g is continuous at x provided $g(x) \neq 0$.

A function is said to be **continuous in a set** if it is continuous at every point of the set.

Theorem 4.9. *(a) Let f be defined in some E and continuous at $x \in E$ and Φ a function on some set containing $f(x)$ and continuous at $f(x)$. Then $\Phi \circ f$ is continuous at x, that is,*

$$\lim_{y\to x}\Phi(f(y)) = \Phi(f(x))\ .$$

(b) Let f be continuous on E and Φ continuous on E_1 where $f(E) \subset E_1$. Then $\Phi \circ f$ is continuous on E.

Now let us review some frequently used functions. First of all, a linear function is of the form

$$f(x) = \sum_{j=1}^{n} \alpha_j x_j + \beta \ , \qquad \alpha_j, \beta \in \mathbb{R} \ .$$

It is a constant function when all α_j's vanish. It is clear from the definition that all linear functions are continuous in $\mathbb{R}^n$. Next, a quadratic function or a quadratic polynomial is of form

$$f(x) = \sum_{j,k=1}^{n} \alpha_{jk} x_j x_k + \sum_{j=1}^{n} \beta_j x_j + \gamma \ , \qquad \alpha_{ij}, \beta_j, \gamma \in \mathbb{R} \ ,$$

where at least one of the $\alpha_{jk} \neq 0$. To describe a polynomial we introduce the notation

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \ , \quad \alpha = (\alpha_1, \cdots, \alpha_n) \ .$$

A polynomial $P(x)$ is a function in the form

$$P(x) = \sum_{1 \leq |\alpha| \leq m} c_\alpha x^\alpha \ , \quad c_\alpha \in \mathbb{R}, \quad |\alpha| = \alpha_1 + \cdots \alpha_n.$$

The degree of a polynomial is the highest combined power with non-zero coefficient appearing in its expression. For instance, in the following polynomials

$$P_1(x,y) = x - 6xy + x^2 y - 17y^3 \ , \quad P_2(x,y,z) = 2 - x - y + xyz^2 - x^3 z \ ,$$

their degrees are 3 and 4 respectively. A repeated application of Theorem 4.8 shows that all polynomials are continuous in $\mathbb{R}^n$.

Next, a rational function $R(x)$ is the quotient of two polynomials, that is,

$$R(x) = \frac{P(x)}{Q(x)} \ , \text{ where } P, Q \text{ are polynomials } , Q \neq 0 \ .$$

The function R is defined in the set where Q is non-zero, that is, $\{x \in \mathbb{R}^n : \ Q(x) \neq 0\}$. Consider the examples

$$R_1(x,y) = \frac{x - xy + x^2 y}{x - y + 1} \ , \quad R_2(x,y,z) = \frac{2 - xyz}{1 + x^2 + y^2 + z^2} \ , \quad R_3(x,y) = \frac{x^2 y - xy^2}{x} \ .$$

The domain of definition of R_1 is the set $\{(x,y) : \ x - y + 1 \neq 0\}$. On the other hand, as the numerator of R_2 never vanish, its domain of definition is $\mathbb{R}^3$. Finally, the domain of definition of R_3 is $\{(x,y) : \ x \neq 0\}$. We observe that after cancelation, the expression defining R_3 is equal to $xy - y^2$ which is well-defined in the entire space. We may say R_3

extends to be a function in $\mathbb{R}^2$. However, the domain of definition of R_3, as it stands, is still the open set $\{(x, y) : \ x \neq 0\}$.

In calculus we encountered many commonly used continuous functions, some of which are

- The **power function** $t \mapsto t^a, a \geq 1$, whose domain of definition is $[0, \infty)$.
- The **radical function** $t \mapsto t^a, a \in (0, 1)$ whose domain of definition is $[0, \infty)$.
- The **exponential function** $t \mapsto e^t$ whose domain of definition is $\mathbb{R}$.
- The **logarithmic function** $t \mapsto \log t$ whose domain of definition is $(0, \infty)$.
- The **trigonometric functions** $\sin t, \cos t$ whose domain of definition is $\mathbb{R}$ and $\tan t$ whose domain of definition is $\mathbb{R} \setminus \{(n + \frac{1}{2})\pi, n \in \mathbb{Z}\}$.
- The inverse trigonometric functions $\arcsin t, \arccos t$, and $\arctan t$ whose domains of definition will be described when needed.
- The **absolute value function** $t \mapsto |t|$ whose domain of definition is $\mathbb{R}$.

We will call these functions **elementary functions**. Using them as Φ in Theorem 4.9, we can form compositions among these functions and the rational functions to produce many examples of continuous functions. Let us consider two examples.

Example 4.6. Consider

$$\log \left(\frac{xy}{x^2 + y^2} \right) \ .$$

We need to determine the set where this formula defines a function before considering its continuity. Indeed, the logarithmic function is defined and continuous on $(0, \infty)$. Therefore, this function is well-defined in the open set $D \equiv \{(x, y) : x, y > 0, \text{ or } x, y < 0\}$, that is, the first and the third quadrants. By Theorem 4.9 it is continuous in D.

Very often we encounter the situation where a function is given in an explicit form but its domain of definition is not specified. When this happens, it is understood that the domain of definition is taken to be the largest set in which the formula makes sense. It may be called the "natural domain" of the function.

Example 4.7. Determine the natural domain of the function given by the formula

$$\sqrt{\sin(x^2 + y^2)} \ .$$

Well, we note that the square root function is defined only on $[0, \infty)$, so we need to ensure $\sin(x^2 + y^2) \geq 0$. After examining the sign of the sine function, we conclude that the natural domain of this formula is

$$D \equiv \{(x, y) : \ x^2 + y^2 \in [2n\pi, (2n+1)\pi], \ n \geq 0 \ \}.$$

By Theorem 4.9 it is continuous in E.

The following result describes the preimages of open intervals under a continuous function are open sets. This provides an efficient way to determine whether a set is open or not .

Theorem 4.10. *Let f be continuous in $\mathbb{R}^n$. Then for every $c \in \mathbb{R}$, the sets*

$$\{x : \ f(x) < c\}, \quad \{x : \ f(x) > c\} \ ,$$

are open, and the sets

$$\{x : \ f(x) \leq c\}, \quad \{x : \ f(x) \geq c\} \ ,$$

are closed. As a result, the level set $\{x : \ f(x) = c\}$ is closed.

Proof. * Let $F = \{x : \ f(x) \geq c\}$. Let $\{x_k\}$ be a sequence in F and $x_k \to x$, so $f_k(x) \geq c$ for all n. Letting $k \to \infty$,

$$f(x) = \lim_{k \to \infty} f(x_k) \geq c \ ,$$

by continuity. We conclude that $x \in F$, that is, F is a closed set. Its complement $\{x : \ f(x) < c\}$ is therefore open. The other case follows after replacing f and c by $-f$ and $-c$ respectively. Finally, observe that

$$\{x : \ f(x) = c\} = \{x : \ f(x) \leq c\} \cap \{x : \ f(x) \geq c\}$$

and the intersection of closed sets is closed, this level set is a closed set. □

So far we have been considering the continuity of real-valued functions. The concept can be extended to any function defined in a subset of some $\mathbb{R}^n$ to $\mathbb{R}^m$. Indeed, let $f(x) = (f_1(x), f_2(x), \cdots, f_m(x))$ be a vector-valued function where f_j is the j-th component of f. Then f is said to be **continuous** at x if and only if f_j is continuous at x for all j. Alternatively, one can say that

$$f : E \subset \mathbb{R}^n \to \mathbb{R}^m \ ,$$

is continuous at $x \in E$ if, for every $\varepsilon > 0$, there is some δ such that

$$|f(y) - f(x)| < \varepsilon \ , \quad \forall y \in B_\delta(x) \ .$$

With this definition, Theorem 4.7, Theorem 4.8(a) and Theorem 4.9 all hold in the vector-valued setting. In particular, Theorem 4.9(b) asserts that composition of functions preserves continuity.

Comments on Chapter 4.

Examples of functions of several variables can be found in the references listed below. Study some of them, and especially familiarize yourself with their graphs.

Supplementary Readings

2.3-2.7 in [Au]. 14.1 and 14.2 in [Thomas].